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EXTENSIONS OF MEYERS–ZIEMER RESULTS

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ABSTRACT

Let $p \in (1, +\infty)$ and $s \in (0, +\infty)$ be two real numbers, and let $H_p^s(\mathbb{R}^n)$ denote the Sobolev space defined with Bessel potentials. We give a class $\mathcal{A}$ of operators, such that $B_{s,p}$-almost all points of $\mathbb{R}^n$ are Lebesgue points of $T(f)$, for all $f \in H_p^s(\mathbb{R}^n)$ and all $T \in \mathcal{A}$ ($B_{s,p}$ denotes the Bessel capacity); this extends the result of Bagby and Ziemer (cf. [2], [15]) and Bojarski–Hajlasz [4], valid whenever T is the identity operator. Furthermore, we describe an interesting special subclass $\mathcal{C}$ of $\mathcal{A}$ ($\mathcal{C}$ contains the Hardy–Littlewood maximal operator, Littlewood–Paley square functions and the absolute value operator T: $f \to |f|$) such that, for every $f \in H_p^s(\mathbb{R}^n)$ and every $T \in \mathcal{C}$, $T(f)$ is quasiuniformly continuous in $\mathbb{R}^n$; this yields an improvement of the Meyers result [10] which asserts that every $f \in H_p^s(\mathbb{R}^n)$ is quasicontinuous. However, $T(f)$ does not belong, in general, to $H_p^s(\mathbb{R}^n)$ whenever $T \in \mathcal{C}$ and $s \geq 1 + 1/p$ (cf. Bourdaud–Kateb [5] or Korry [7]).

1. Introduction

It is well-known that for every function f belonging to the Lebesgue space $L_1^{loc}(\mathbb{R}^n)$, almost every point $x \in \mathbb{R}^n$ is a Lebesgue point of f:

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)|\, dy = 0;$$

here, $\fint_{B(x,r)} u(y)\, dy$ denotes the mean value of u over the ball $B(x,r)$ of $\mathbb{R}^n$.

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One should think of the Bessel capacity –for its definition, see below– as a finer tool than Lebesgue measure (in the sense that a set may have positive capacity, although it has measure zero).

In the case of Sobolev functions, one can do better than above: Bagby and Ziemer (cf. [2], [15]) proved that if $f \in W^{k,p}(\mathbb{R}^n)$ where $k \geq 1$ is a natural number, then $B_{k,p}$-almost all points of $\mathbb{R}^n$ are Lebesgue points of f (here, $B_{k,p}$ denotes the Bessel capacity). This result has been extended by Bojarski and Hajlasz [4] to the fractional Sobolev spaces $H_p^s(\mathbb{R}^n)$, defined by Bessel potentials where $s \in (0, +\infty)$ and $p \in (1, +\infty)$; we recall that $W^{k,p}(\mathbb{R}^n) = H_p^k(\mathbb{R}^n)$, see Stein [11].

The aim of this paper is to give a class $\mathcal{A}$ of operators, such that $B_{s,p}$-almost all points of $\mathbb{R}^n$ are Lebesgue points of $T(f)$, for every $f \in H_p^s(\mathbb{R}^n)$ and every $T \in \mathcal{A}$; this extends the result of Bagby–Ziemer recalled above, valid whenever T is the identity operator (i.e. $T(f) = f$ for every $f \in H_p^s(\mathbb{R}^n)$) with $s \in (0, +\infty)$ and $p \in (1, +\infty)$. Furthermore, we describe (cf. Section 3) an interesting special subclass $\mathcal{C}$ of $\mathcal{A}$ such that, for every $f \in H_p^s(\mathbb{R}^n)$ and every $T \in \mathcal{C}$, $T(f)$ is quasiuniformly continuous in $\mathbb{R}^n$; this yields an improvement of Meyers' result [10] which asserts that every $f \in H_p^s(\mathbb{R}^n)$ is quasicontinuous, and another result of Kinnunen [6] valid for $s = 1$ in the case of the Hardy–Littlewood maximal operator. However, $T(f)$ does not belong, in general, to $H_p^s(\mathbb{R}^n)$ whenever $T \in \mathcal{C}$ and $s \geq 1 + 1/p$: indeed, we consider the following variant of the Hardy–Littlewood maximal operator, defined by means of the Gauss semigroup $(\varphi_t)_{t>0}$ by setting

$$T(f) = \sup_{t>0} |f * \varphi_t(x)|,$$

where $\varphi_t(x) = t^{-n/2}\varphi(x/\sqrt{t})$ and $\varphi(x) = (2\pi)^{-n/2}\exp(-|x|^2/2)$. It is simple to check that $T \in \mathcal{C}$ (see Section 3 for the definition of the subclass $\mathcal{C}$). However, if we choose $f = \partial/\partial_{x_1}\varphi$ which belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, by using the identity $\varphi_t * f = \partial/\partial_{x_1}\varphi_{t+1}$, we obtain $T(f)(x) = |x_1|\varphi(x)$ for every $|x| \leq 1$. According to the fact that $|x_1|$ does not belong locally near the origin to $H_p^s(\mathbb{R}^n)$ as soon as $s \geq 1 + 1/p$ (due to the fact that, in the one-dimensional situation, the characteristic function $\chi_{[-1,1]}$ does not belong to $H_p^{s-1}(\mathbb{R})$ as soon as $s \geq 1 + 1/p$, cf. Strichartz [12]), this completes our claim.

Bessel capacity. Let us recall some facts about the Bessel capacity. It occurs naturally in the study of the deeper properties of the Sobolev functions $H_p^s(\mathbb{R}^n)$. For the capacity theory, we refer to the monographs Adams–Hedberg [1], Maz'ya [8] and Ziemer [16]. For every $s \in (0, +\infty)$, we denote by G_s the kernel of the operator $(I-\Delta)^{-s/2}$; this kernel (cf. Stein [11], page 132) is a positive function and belongs to $L_1(\mathbb{R}^n)$. Classically, the Bessel capacity of a set $E \subset \mathbb{R}^n$ is defined by setting

$$B_{s,p}(E) = \inf\{\|u\|_p^p\colon G_s * u \geq 1 \text{ on } E,\ u \geq 0\}.$$

The sets of zero $B_{s,p}$-capacity have Hausdorff dimension less than or equal to $n-sp$. Precisely, we have the following results (cf. Meyers [10], Theorems 20 and 21):

– If $sp < n$, then

$$H_{n-sp}(E) < \infty \Longrightarrow B_{s,p}(E) = 0,$$
$$B_{s,p}(E) = 0 \Longrightarrow \forall \varepsilon > 0,\ H_{n-sp+\varepsilon}(E) = 0$$

(here, H_α denotes the Hausdorff measure).

– If $sp > n$, then there exists a constant $C > 0$ such that

$$\forall E \subset \mathbb{R}^n, E \neq \emptyset \Longrightarrow B_{s,p}(E) > C.$$

A function f belongs to $H_p^s(\mathbb{R}^n)$, where $p \in (1,+\infty)$ and $s \in (0,+\infty)$ if, and only if, there exists a function $g \in L_p(\mathbb{R}^n)$ such that $f = G_s * g$; one sets $\|f\|_{H_p^s} = \|g\|_p$.

The class $\mathcal{A}$. An operator T (possibly non-linear), defined on $L_p(\mathbb{R}^n)$, belongs to $\mathcal{A}$ if, and only if, there exists a constant $C(T) > 0$ such that the following conditions hold:

(i) for every $u \in L_p(\mathbb{R}^n)$, $\|T(u)\|_p \leq C(T)\ \|u\|_p$;

(ii) for every $u \in L_p(\mathbb{R}^n)$, $|T(G_s * u)| \leq C(T)|G_s * T(u)|$ a.e.;

(iii) for every $u_1, u_2 \in L_p(\mathbb{R}^n)$, $|T(u_1) - T(u_2)| \leq C(T)|T(u_1 - u_2)|$ a.e.;

(iv) for every $u \in \mathcal{S}(\mathbb{R}^n)$ and for every $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that, for every $x \in \mathbb{R}^n$, $\mathcal{M}^\#_{R_\varepsilon}(Tu)(x) < \varepsilon$, where we set

$$\mathcal{M}^\#_R(u)(x) = \sup_{0<r<R} ⨍_{B(x,r)} |u(y) - u(x)|\, dy.$$

Remark 1: Throughout this paper, for $u \in H_p^s(\mathbb{R}^n)$, the function Tu is defined everywhere by selecting a representative, essentially by an everywhere convergent limiting process of a sequence of continuous real-valued functions. This is best illustrated by using the following formula:

$$\widetilde{Tu}(x) = \limsup_{r\to 0} ⨍_{B(x,r)} Tu(y)\, dy.$$

In what follows, as a rule, we identify $\widetilde{Tu}$ with Tu and omit the tilde sign.

Remark 2: Note that an operator T satisfies the condition **(iv)** if, for every $u \in \mathcal{S}(\mathbb{R}^n)$, Tu is uniformly continuous on $\mathbb{R}^n$, and even more so if $Tu \in Lip_\eta(\mathbb{R}^n)$ for some $0 < \eta \leq 1$; we recall that $f \in Lip_\eta(\mathbb{R}^n)$ if, and only if, for every $x, y \in \mathbb{R}^n$, $|f(x) - f(y)| \leq C\ |x-y|^\eta$ for some constant C.

Now, let $f \in H_p^s(\mathbb{R}^n)$ with $p \in (1, +\infty)$ and $s \in (0, +\infty)$; for every $T \in \mathcal{A}$, we have the following results.

THEOREM 1: *For every real number $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset \mathbb{R}^n$ such that $B_{s,p}(U_\varepsilon) < \varepsilon$ and*

$$\lim_{R \to 0} \mathcal{M}_R^{\#}[T(f)] = 0,$$

uniformly on $\mathbb{R}^n \backslash U_\varepsilon$.

COROLLARY 1: *$B_{s,p}$-almost all points of $\mathbb{R}^n$ are Lebesgue points of $T(f)$.*

Let us mention an interesting special case of the preceding situation, namely, the Littlewood–Paley square function $f \to S(f)$:

Let ψ be a real integrable function on $\mathbb{R}^n$ satisfying the following conditions:

$$\int_{\mathbb{R}^n} \psi(x)\ dx = 0, \quad |\psi(x)| \leq C|x|^{-n-\epsilon} \text{ and } \int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)|\ dx \leq C|y|^\epsilon,$$

for some fixed real number $\epsilon > 0$. A Littlewood–Paley square function type, corresponding to ψ, is

$$S(f)(x) = S_\psi(f)(x) = \sqrt{\int_0^{+\infty} |f * \psi_t(x)|^2 \frac{dt}{t}},$$

where $\psi_t(x) = t^{-n}\psi(x/t)$. We recall that S is bounded in the Lebesgue space $L_p(\mathbb{R}^n)$ for every $p \in (1, +\infty)$; this result is obtained as a consequence of results concerning "Hilbert-valued" Calderón–Zygmund operators (cf. Benedek, Calderón and Panzone [3]). Obviously, such an operator S satisfies the conditions **(i)**, **(ii)** –because $G_s \geq 0$– and **(iii)**. To check the condition **(iv)**, we use the following estimate (cf. [3], page 506):

$$\sup_{\xi \in \mathbb{R}^n} \int_0^{+\infty} |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} < +\infty$$

($\widehat{\psi}$ denotes the Fourier transform of ψ), which yields that $S(u)$ belongs to $Lip_1(\mathbb{R}^n)$ when $u \in \mathcal{S}(\mathbb{R}^n)$.

Well-known examples of the square function of Littlewood–Paley are as follows.

Example 1: Let $(P_t)_{t>0}$ be the Poisson semi-group defined on the upper half space $\mathbb{R}^n \times (0, +\infty)$ by

$$P_t(x) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}.$$

Put

$$\psi(x) = \Big(\frac{\partial}{\partial t} P_t(x)\Big)_{t=1}.$$

Then, $S_\psi(f)$ is the Littlewood–Paley g-function.

Example 2: Consider the Haar function ψ on $\mathbb{R}$: $\psi = 1_{[-1,0]} - 1_{[0,1]}$. Then, $S_\psi(f)$ is the Marcinkiewicz integral

$$\mu(f)(x) = \Big(\int_0^{+\infty} |F(x+t) + F(x-t) - 2F(x)|^2 \frac{dt}{t^3}\Big)^{1/2},$$

where $F(x) = \int_0^x f(y)dy$.

Beside Littlewood–Paley square functions, $\mathcal{A}$ contains the Hardy–Littlewood maximal operator $\mathcal{M}$ which is defined, for every locally integrable function f and every $x \in \mathbb{R}^n$, by setting

$$\mathcal{M}(f)(x) = \sup_{r>0} ⨍_{B(x,r)} |f(y)|\, dy.$$

Indeed, the celebrated theorem of Hardy, Littlewood and Wiener (cf. Stein [11]) asserts that $\mathcal{M}$ is bounded in $L_p(\mathbb{R}^n)$ for all $1 < p \leq \infty$. So $\mathcal{M}$ satisfies **(i)**, **(ii)** and **(iii)**; the condition **(iv)** follows easily from the inequality

$$|\mathcal{M}(f)(x+h) - \mathcal{M}(f)(x)| \leq \mathcal{M}(f(\cdot + h) - f(\cdot))(x)$$

and the boundedness of $\mathcal{M}$ in $L_\infty(\mathbb{R}^n)$.

In the linear case, the class $\mathcal{A}$ contains the multiplier operator with a fixed bounded uniformly continuous function m (i.e. $T(f) = m\ f$) and some pseudo-differential operators like the Hilbert transform $f \to H(f)$:

$$H(f)(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \frac{f(y)}{x-y}\, dy (x \in \mathbb{R}).$$

We recall that H is bounded in $L_p(\mathbb{R})$, for every $p \in (1, +\infty)$, and bounded in $Lip_\alpha(\mathbb{R})$ for every $0 < \alpha < 1$, cf. Meyer [9], page 328.

2. Proofs

2.1. An elementary approach for $(n/p < 1 + 1/p)$. In this subsection, we describe, by means of the Calderón–Zygmund operators, a subclass $\mathcal{B}$ of the class $\mathcal{A}$ for which we shall give a particular proof of our result (i.e. Corollary 1) under the restriction $(n/p < 1 + 1/p)$.

We say that a linear operator U is of Calderón–Zygmund type (**CZ** operator, for short), if there exists a real separable Hilbert space $\mathcal{H}$ such that U is bounded from $L^r(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n, \mathcal{H})$ for some fixed real number $r \in (1, +\infty)$, and if there exists a strongly measurable $\mathcal{H}$-valued kernel K defined on $\mathbb{R}^n$, locally integrable outside the origin such that

– if f is any scalar continuous function with compact support and if x does not belong to $\mathrm{supp}(f)$, then

$$U(f)(x) = \int_{\mathbb{R}^n} f(y)K(x-y)dy;$$

– (Hörmander's condition) there exists a constant $C \geq 0$ such that

$$\forall y \in \mathbb{R}^n, \quad \int_{|x|>2|y|} |K(x-y) - K(x)|_{\mathcal{H}} dx \leq C.$$

Now, an operator T belongs to $\mathcal{B}$ if it satisfies the condition **(iv)** and, for every $f \in L_p(\mathbb{R}^n)$, $T(f) = |U(f)|_{\mathcal{H}}$ for some **CZ** operator U such that the following conditions hold:

- the operator U commutes with translations (i.e. $\tau_\alpha T = T\tau_\alpha$ for every $\alpha \in \mathbb{R}^n$, where $\tau_\alpha f(x) = f(x-\alpha)$) and with Littlewood–Paley operators $(\Delta_k)_{k\in\mathbb{Z}}$; here, $\Delta_k(f) = f * \psi_k$ and $\psi_k(x) = 2^{nk}\psi(2^k x)$, where $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that the Fourier transform $\widehat{\psi}$ of ψ satisfies $\sum_{j\in\mathbb{Z}} \widehat{\psi}(2^{-j}\xi) = 1$ for every $\xi \neq 0$ and $\mathrm{supp}(\widehat{\psi}) \subset \{\xi \in \mathbb{R}^n \colon 1/2 \leq |\xi| \leq 2\}$. These conditions imply that T is bounded on $H_p^s(\mathbb{R}^n)$ for every $p \in (1, +\infty)$ and $s \in [0, 1+1/p)$, see [7]; the condition $s < 1 + 1/p$ is optimal: there exists such an operator T and a function $f \in C_c^\infty(\mathbb{R}^n)$ such that, for every $p \in (1, +\infty)$ and every $s \geq 1 + 1/p$, $T(f)$ does not belong to $H_p^s(\mathbb{R}^n)$.
- there exists $0 < \eta < 1$ such that, for every $f \in L_p(\mathbb{R}^n) \cap C^\eta(\mathbb{R}^n)$, $T(f)$ is continuous (we recall that $f \in C^\eta(\mathbb{R}^n)$ if, and only if, $f \in L_\infty \cap Lip_\eta$).

Obviously, such an operator T belongs to $\mathcal{A}$. Let us mention that $\mathcal{B}$ contains some Littlewood–Paley square functions. The boundedness of Littlewood–Paley square functions on $Lip_\eta(\mathbb{R}^n)$ has been discussed by Wang [14].

Now, under these assumptions, for every $T \in \mathcal{B}$, if $n/p < 1 + 1/p$ and $f \in$

$H_p^s(\mathbb{R}^n)$ where $s \in (0, +\infty)$ and $p \in (1, +\infty)$, then $B_{s,p}$-almost all points of $\mathbb{R}^n$ are Lebesgue points of $T(f)$.

Indeed, on the one hand, if $0 < s \leq n/p$, the assumption $n/p < 1 + 1/p$ yields that $0 < s < 1 + 1/p$, so $T(f) \in H_p^s(\mathbb{R}^n)$ (cf. Korry [7]). Therefore, according to the result of Bojarski and Hajlasz [4] above, we obtain our claim in this case. On the other hand, if $s > n/p$, we use the embedding theorem $H_p^s(\mathbb{R}^n) \subset L_p(\mathbb{R}^n) \cap C^{s-n/p}(\mathbb{R}^n)$. So, according to the assumptions above, $T(f)$ is continuous. Consequently, all points of $\mathbb{R}^n$ are Lebesgue points of $T(f)$. This completes the proof of our claim. ∎

2.2. Proof of Theorem 1. First of all, we shall prove the following weak-type estimate which is the key tool in the proof of our results:

$$\text{(1)} \qquad \exists C > 0,\ \forall \lambda > 0,\ B_{s,p}(\{\mathcal{M}[T(f)] > \lambda\}) \leq C\lambda^{-p}\|f\|_{H_p^s}^p.$$

Proof of the estimate (1): Let $f \in H_p^s(\mathbb{R}^n)$; there exists $g \in L_p(\mathbb{R}^n)$ such that $f = G_s * g$ and $\|f\|_{H_p^s} = \|g\|_p$. We apply the condition **(ii)**, given in the definition of the class $\mathcal{A}$, and the fact that the Hardy–Littlewood maximal operator $\mathcal{M}$ is increasing on the positive cone of $L_p(\mathbb{R}^n)$, namely,

$$\{f \in L^p(\mathbb{R}^n)\colon f(x) \geq 0 \text{ almost everywhere}\}.$$

Hence, we obtain

$$\mathcal{M}[T(f)] \leq G_s * \mathcal{M}[C(T)T(g)].$$

The definition of the Bessel capacity $B_{s,p}$ leads to

$$\begin{aligned} B_{s,p}(\{\mathcal{M}[T(f)] > \lambda\}) \leq& B_{s,p}(\{G_s * \mathcal{M}[C(T)T(g)] > \lambda\}) \\ \leq& \Big\|\mathcal{M}\Big[\frac{C(T)}{\lambda}T(g)\Big]\Big\|_p^p \leq C\lambda^{-p}\|f\|_{H_p^s}^p. \end{aligned}$$

The last inequality follows from the boundedness of the operators $\mathcal{M}$ and T in the Lebesgue space $L_p(\mathbb{R}^n)$.

Proof of Theorem 1: Now, we are ready to prove our Theorem 1. This second part of the proof of Theorem 1 closely follows the proof of Theorem 8 in [4]. Let $\varepsilon > 0$ be a real number. There exists $f_\varepsilon \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f_\varepsilon - f\|_{H_p^s}^p \leq C^{-1}\varepsilon^{p+1}$, where C is the constant which is given in the weak-type estimate (1). For every real number $R > 0$,

$$\begin{aligned} \mathcal{M}_R^{\#}T(f) \leq& \mathcal{M}_R^{\#}T(f_\varepsilon) + \mathcal{M}_R^{\#}[T(f) - T(f_\varepsilon)] \\ \leq& \mathcal{M}_R^{\#}T(f_\varepsilon) + 2\mathcal{M}[T(f) - T(f_\varepsilon)] \\ \leq& \mathcal{M}_R^{\#}T(f_\varepsilon) + 2C(T)\mathcal{M}[T(f - f_\varepsilon)]. \end{aligned}$$

According to the condition **(iv)**, given in the definition of the class $\mathcal{A}$, there exists R_ε such that $\mathcal{M}^{\#}_{R_\varepsilon}T(f_\varepsilon) < \varepsilon$. Hence, the weak estimate (1) yields

$$B_{s,p}(\{\mathcal{M}^{\#}_{R_\varepsilon}T(f) > (1+2C(T))\varepsilon\}) \leq B_{s,p}(\{\mathcal{M}[T(f-f_\varepsilon)] > \varepsilon\}) \leq \varepsilon.$$

Now, let $i \in \mathbb{N}$, $\varepsilon_i = 2^{-i}\varepsilon$ and $R_i = R_{\varepsilon_i}$; then for every $i \in \mathbb{N}$, we have

$$B_{s,p}(\{\mathcal{M}^{\#}_{R_i}T(f) > \varepsilon_i\}) \leq \frac{\varepsilon}{1+2C(T)}2^{-i}.$$

Let

$$V = \bigcup_{i\in\mathbb{N}}\{\mathcal{M}^{\#}_{R_i}T(f) > \varepsilon_i\};$$

therefore,

$$B_{s,p}(V) \leq \sum_{i\in\mathbb{N}} B_{s,p}(\mathcal{M}^{\#}_{R_i}T(f) > \varepsilon_i) \leq \frac{\varepsilon}{1+2C(T)}.$$

Finally, we use the fact (cf. Ziemer [16]) that

$$B_{s,p}(V) = \inf\{B_{s,p}(U) : U \supset V,\ U \text{ open}\}; \tag{2}$$

so there exists an open set $U_\varepsilon \supset V$ such that $B_{s,p}(U_\varepsilon) < \varepsilon$. This completes the proof of Theorem 1. ∎

Before we conclude, we would like to study the quasicontinuity of a subclass $\mathcal{C}$ of the class $\mathcal{A}$.

3. Quasicontinuity

An important and very useful property of the Sobolev space $H^s_p(\mathbb{R}^n)$ is the fact that every element of this space, viewed as an equivalence class of functions coinciding a.e., has a distinguished representative which is quasicontinuous with respect to some suitable capacity; see Meyers [10]. This means, in particular, that the representative is defined more often than almost everywhere with respect to Lebesgue measure, namely, outside a set of zero capacity, where the exceptional sets are measured by Bessel capacity $B_{s,p}$ if $p \in (1,+\infty)$, or by some equivalent capacity. It is thus meaningful to talk about the values of a Sobolev function on "small" sets with measure zero but with positive capacity. The property of being quasicontinuous may more generally be viewed as the counterpart in potential theory to the familiar Luzin property of measurable functions.

SUBCLASS $\mathcal{C}$. We are interested to extend the Meyers result, recalled above, for a subclass $\mathcal{C}$ of the class $\mathcal{A}$. This class is defined as follows: an operator T belongs to $\mathcal{C}$ if, and only if, T satisfies the previous properties **(i)**, **(ii)** and **(iii)** –given in the definition of the class $\mathcal{A}$– and if, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, the function $T(\phi)$ is uniformly continuous on $\mathbb{R}^n$. It is simple to check that the class $\mathcal{C}$ contains the Hardy–Littlewood maximal operator, Littlewood–Paley square functions and the absolute value operator $T: f \to |f|$.

Let us first recall the quasicontinuity notion with respect to the Bessel capacity. A function f is $B_{s,p}$-quasicontinuous (resp. quasiuniformly continuous) in $\mathbb{R}^n$, if for every $\varepsilon > 0$ there exists an open set U_ε such that $B_{s,p}(U_\varepsilon) \leq \varepsilon$ and the restriction of f to $\mathbb{R}^n \backslash U_\varepsilon$ is finite and continuous (resp. uniformly continuous).

THEOREM 2: *Let $p \in (1, +\infty)$, $s \in (0, +\infty)$, $T \in \mathcal{C}$ and $f \in H_p^s(\mathbb{R}^n)$. Then, $T(f)$ is $B_{s,p}$-quasiuniformly continuous.*

Proof of Theorem 2: Let $(\phi_i)_{i\geq 1}$ be a sequence of functions $\phi_i \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi_i \to f$ in $H_p^s(\mathbb{R}^n)$. By the weak-type inequality (1), there exists a set F with $B_{s,p}(F) = 0$ so that $\mathcal{M}(Tf)$ is finite in $\mathbb{R}^n \backslash F$. We choose a subsequence, which is denoted again by $(\phi_i)_{i\geq 1}$, such that

$$\|f - \phi_i\|_{H_p^s}^p \leq C^{-1} 4^{-ip}.$$

Consider the sequence $(E_i)_{i\geq 1}$, where

$$E_i = \{x \in \mathbb{R}^n \backslash F: \mathcal{M}(T(f - \phi_i)) > 2^{-i}\}.$$

Using the weak-type inequality (1), we obtain

$$B_{s,p}(E_i) \leq 2^{ip} C \|f - \phi_i\|_{H_p^s}^p \leq 2^{-ip}.$$

On the one hand, for every $j \geq 1$, set $F_j = \bigcup_{i\geq j} E_i$; then, by subadditivity

$$B_{s,p}(F_j) \leq \sum_{i\geq j} 2^{-ip} = \frac{2^{-jp}}{1 - 2^{-p}}.$$

Consequently, $\lim_{j\to+\infty} B_{s,p}(F_j) = 0$. Moreover, for every $x \in \mathbb{R}^n \backslash F_j$, we have

$$\begin{aligned} |Tf(x) - T\phi_i(x)| \leq & T(f - \phi_i)(x) \\ \leq & \mathcal{M}(T(f - \phi_i))(x) \leq 2^{-i} \end{aligned}$$

whenever $i \geq j$.

On the other hand, let $\varepsilon > 0$ and choose j_ε such that $B_{s,p}(F_{j_\varepsilon}) \leq \varepsilon/2$ and $2^{-j_\varepsilon} \leq \varepsilon$. Then, according to (2), there exists an open set $U_\varepsilon \supset F_{j_\varepsilon}$ such that $B_{s,p}(U_\varepsilon) \leq \varepsilon$. Therefore, for every $x \notin U_\varepsilon$ and every $j \geq j_\varepsilon$, we have

$$|Tf(x) - T\phi_i(x)| \leq 2^{-j_\varepsilon} \leq \varepsilon$$

whenever $i \geq j_\varepsilon$, which shows that the convergence is uniform in $\mathbb{R}^n \setminus U_\varepsilon$. As a uniform limit of the uniformly continuous functions $(T\phi_i)_{i\geq 1}$, the function $T(f)$ is uniformly continuous on $\mathbb{R}^n \setminus U_\varepsilon$; hence $T(f)$ is $B_{s,p}$-quasiuniformly continuous. ∎

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